This exercise sheet deals with the properties of Bose-Einstein condensates in random potentials, with an emphasis on the effects of mean-field interactions.

This exercise is directly inspired by the following paper: L. Sanchez-Palencia, Phys. Rev. A 74, 053625 (2006).

A. Introduction: Thomas-Fermi regime

We consider a Bose-Einstein condensate with chemical potential $\mu$ subject to a random potential $V(r)$. Interactions between the atoms are described by the coupling constant $g$. The random potential is a speckle pattern with correlation length $\sigma$ (the typical size of potential variations) and an average value $\bar{V}$ (also called the disorder strength). The probability distribution of the speckle potential is $p(V) = \frac{1}{\bar{V}} e^{V/\bar{V}}$. We first suppose that the correlation length of the random potential is large, such that the local density approximation is valid.

1. In one dimension, use the rare-region argument to prove that for arbitrarily small values of $\bar{V}$, at the thermodynamic limit the condensate is composed of several disconnected condensates separated by potential barriers.

2. Can one generalise this to 2D and 3D?

3. Calculate the average density as a function of disorder strength and the fraction of space left empty.

B. Smoothing of the potential due to interactions

We now focus on the case where the correlation length is arbitrary, but the disorder is very weak such that a perturbation approach is valid.

4. Write the Gross-Pitaevskii equation for the condensate in the presence of the potential. Express the condensate wave function as $\psi_0 + \delta \psi$, with $\delta \psi$ a correction of order $\bar{V}$ and show that $\delta \psi$ obeys the equation

$$-\frac{\hbar^2}{2m} \Delta \delta \psi - [\mu - 3g\psi_0^2] \delta \psi = -V(r)\psi_0$$

5. Fourier transform this expression and express the solution as

$$\delta \psi = -\int dr G(r-r') \frac{V(r)\psi_0}{2\mu}$$

with $G$ the a convolution Kernel. Show that in any dimensions, this Kernel is a peaked function of width $\xi$, the healing length of the condensate and give a physical interpretation.

6. Show that for a potential with long correlation length, this expression reduces to the usual Thomas-Fermi approximation. Using this insight, show that the above expression amounts to a Thomas-Fermi condensate placed in a renormalised potential $\tilde{V}$ obtained by convolution with the Kernel.
C. Bonus: smoothing of the potential due to the wave nature of particles

This part is directly inspired by the following paper: B. Shklovskii, Semiconductors 42, 909 (2008).

The smoothing above results from interactions between the particles. However, one can also introduce a smoothing of the potential at the level of the single particle (usually much weaker than the previous one for the typical BEC experiments). We now present a qualitative discussion of the static behaviour of a particle in a disorder and introduce the important notion of Larkin length.

We consider a weak random such that \( \bar{V} \ll \frac{\hbar^2}{m\sigma^2} \) (the energy \( \frac{\hbar^2}{m\sigma^2} \) plays the role of the recoil energy for atoms in optical lattices). In this part of the exercise, we suppose that the random potential is such that \( \langle V \rangle = 0 \) and the strength of the disorder is measured by its variance \( \delta V^2 = V^2 \).

8. Using the Heisenberg principle, show that most of the individual modulations of the potential do not support any bound state. The states that do not support any bound state do not contribute to the static properties of the cloud, such as the density, compressibility etc (however, they are vital to the existence of Anderson localisation !)

9. Consider now a volume \( L^d \), in \( d \) dimensions. This volume contains typically \( (\sigma/L)^d \) independent "fluctuations" of the potential. Supposing that only the wells of size \( > L \) can support a bound state, show that the potential fluctuations \( v(L) \) among the wells that support a bound state of size \( L \) are such that

\[
\frac{v(L)}{\bar{V}} = \left( \frac{\sigma}{L} \right)^{d/2}.
\]

Hint: many independent realisations of potential wells will follow the Poisson distribution such that the variance is proportional to the number of tries.

10. We now introduce the typical size \( l \) of a bound state. Use Heisenberg’s principle to evaluate the typical kinetic energy of such a state. By requiring that the potential modulation that binds the particle has a depth of the order of its kinetic energy, show that

\[
\frac{l}{\sigma} \sim \left( \frac{\hbar^2}{m\sigma^2V} \right)^{d-1}.
\]

The length \( l \) is called the Larkin length, and characterises the density of bound states. The effective disorder strength relevant for the static properties is thus reduced by an amount \( \left( \frac{\sigma}{L} \right)^{d/2} \).

Note that this smoothing is analogous to the well known effect that is a lightwave passes through a set of impurities much denser than the wavelength, it will completely ignore the impurities !.